Math 1A - So you think you can slant (asymptote)?

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In our magical journey of graphing functions, we encountered a completely new concept, that of a **slant asymptote**. Let's first define such a concept!

1 Definition of a slant asymptote

Definition. *A line $y = ax + b$ is a* ***slant asymptote to the graph of*** f ***at*** ∞ *if*

$$\lim_{x \to \infty} f(x) - (ax + b) = 0$$

Similarly for $-\infty$

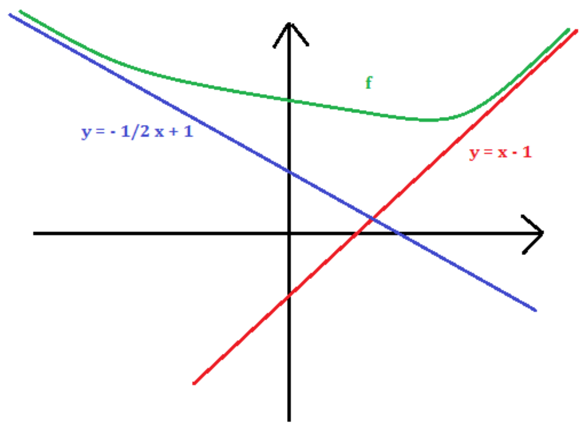
Definition. *A line $y = ax + b$ is a* ***slant asymptote to the graph of*** f ***at -*** ∞ *if*

$$\lim_{x \to -\infty} f(x) - (ax + b) = 0$$

If you don't like a and b, just think of $2x+3$, or $x-1$, or whatever your two favorite numbers are!

This concept is easier understood with a picture! (see next page)

1A/Handouts/Slant.png



Notice that, as $x \to \infty$, the graph of f 'approaches' the line $y = x - 2$, so $y = x - 1$ is a slant asymptote to the graph of f at ∞. Similarly, as $x \to -\infty$, the graph of f 'approaches' the line $y = -\frac{x}{2} + 1$, so $y = -\frac{x}{2} + 1$ is a slant asymptote to the graph of f at $-\infty$!

What's the point, really? Basically, we're saying that as x becomes very large, the graph of f looks very much like $y = x - 1$, which tells us a great deal about the graph of f.

Also, note that the graph of f **CAN** cross its slant asymptote, and it **CAN** approach this asymptote from below! (the picture is a little bit misleading in this sense).

2 How can we show that a given line is a slant asymptote?

One sample problem would be the following:

Problem. *Show that $y = x + 2$ is a slant asymptote to the graph of $f(x) = \sqrt{x^2 + 4x}$*

This is the easiest problem you can get about this topic! First, you need to decide whether the above line is a slant asymptote at ∞ or $-\infty$, and this requires 2 seconds of thinking: Notice that $y = x + 2$ goes to $-\infty$ as $x \to -\infty$, so it **can't** be a slant asymptote at $-\infty$ because f goes to ∞ as $x \to -\infty$! So, if $y = x + 2$ is a slant asymptote, then it must be a slant asymptote **at** ∞!

Now the rest is easy, because we just need to show $\lim_{x\to\infty} \sqrt{x^2+4x}-(x+2) = 0$. A calculation using 'conjugate forms' shows:

$$\begin{aligned}
\lim_{x\to\infty} \sqrt{x^2+4x}-(x+2) &= \lim_{x\to\infty} \left(\sqrt{x^2+4x}-(x+2)\right)\frac{\sqrt{x^2+4x}+(x+2)}{\sqrt{x^2+4x}+(x+2)} \\
&= \lim_{x\to\infty} \frac{x^2+4x-(x+2)^2}{\sqrt{x^2+4x}+(x+2)} \\
&= \lim_{x\to\infty} \frac{x^2+4x-x^2-4x-4}{\sqrt{x^2+4x}+(x+2)} \\
&= \lim_{x\to\infty} \frac{-4}{\sqrt{x^2+4x}+(x+2)} \\
&= 0
\end{aligned}$$

And so, we showed that $y = x+2$ is a slant asymptote to the graph of f at ∞. As an exercise, show that $y = -x-2$ is a slant asymptote to the graph of f at $-\infty$.

3 How can we find slant asymptotes?

There is a wonderful standard procedure to find slant asymptotes, and it is also useful to show that a graph cannot have a slant asymptote! It is based on the following fact:

Suppose $y = ax+b$ is a slant asymptote to f at ∞. Then:

$$\lim_{x\to\infty} f(x)-(ax+b) = 0$$

Now, dividing both sides by x, we get:

$$\lim_{x\to\infty} \frac{f(x)}{x} - \frac{ax+b}{x} = 0$$

That is:

$$\lim_{x\to\infty} \frac{f(x)}{x} - a + \frac{b}{x} = 0$$

However, $\lim_{x\to\infty} \frac{b}{x} = 0$, so we get:

$$\lim_{x\to\infty} \frac{f(x)}{x} - a = 0$$

That is:

$$a = \lim_{x\to\infty} \frac{f(x)}{x}$$

In other words, the slope of the slant asymptote is given by $\boxed{a = \lim_{x\to\infty} \frac{f(x)}{x}}$.

And once you found a, finding b (the $y-$intercept) is even easier, namely:

$$b = \lim_{x\to\infty} f(x) - ax$$

(you get this by adding b to both sides of the equality: $0 = \lim_{x\to\infty} f(x) - (ax + b)$).

And of course, the same thing is true for $-\infty$.

Now let's see this in action!

Problem. *Find the slant asymptote to* $f(x) = \sqrt{x^2 + 4x}$ *at* ∞

This is the same problem as above, except that we're not given the equation of the slant asymptote! Here, we'll try to find it!

First let's find the slope a:

$$\begin{aligned} a &= \lim_{x\to\infty} \frac{\sqrt{x^2+4x}}{x} \\ &= \lim_{x\to\infty} \frac{\sqrt{(x^2)(1+\frac{4}{x})}}{x} \\ &= \lim_{x\to\infty} \frac{|x|\sqrt{1+\frac{4}{x}}}{x} \\ &= \lim_{x\to\infty} \frac{x\sqrt{1+\frac{4}{x}}}{x} \qquad \text{(since } x > 0\text{)} \\ &= \lim_{x\to\infty} \sqrt{1+\frac{4}{x}} \\ &= 1 \end{aligned}$$

And to find b, we get:

$$
\begin{aligned}
b &= \lim_{x\to\infty} \sqrt{x^2+4x} - x \\
&= \lim_{x\to\infty} \frac{x^2+4x-x^2}{\sqrt{x^2+4x}+x} \qquad\qquad \text{(conjugate form)} \\
&= \lim_{x\to\infty} \frac{4x}{\sqrt{x^2+4x}+x} \\
&= \lim_{x\to\infty} \frac{4x}{|x|\sqrt{1+\frac{4}{x}}+x} \\
&= \lim_{x\to\infty} \frac{4x}{x\sqrt{1+\frac{4}{x}}+x} \\
&= \lim_{x\to\infty} \frac{4x}{x\left(\sqrt{1+\frac{4}{x}}+1\right)} \\
&= \lim_{x\to\infty} \frac{4}{\sqrt{1+\frac{4}{x}}+1} \\
&= \frac{4}{2} \\
&= 2
\end{aligned}
$$

Hence, the slant asymptote to f at ∞ is: $\boxed{y = x+2}$ (which is the same answer we found above!)

This procedure is also good to show a function cannot have a slant asymptote!

Problem. *Show that $f(x) = x + \sqrt{x}$ does not have a slant asymptote at ∞*

We'll do a proof by contradiction! Suppose f has a slant asymptote $y = ax + b$. Then we must have:

$$
a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x+\sqrt{x}}{x} = \lim_{x\to\infty} 1 + \frac{1}{\sqrt{x}} = 1
$$

So $y = x + b$.

And then, we get:

$$
b = \lim_{x\to\infty} f(x) - x = \lim_{x\to\infty} x + \sqrt{x} - x = \lim_{x\to\infty} \sqrt{x} = \infty
$$

Which is a contradiction (since b must be finite!).

Hence f cannot have a slant asymptote at ∞.

A couple of other useful remarks:

1. If you're given a function f that is the sum of a linear function and another function, i.e. $f(x) = g(x) + h(x)$, then try $g(x)$ as your guess. For example, if $f(x) = e^x - x = -x + e^x$, then try $y = -x$ as your slant asymptote. Again, **beware** whether it is an asymptote at $+\infty$ or at $-\infty$!!! Here, if you do the 'showing'-part, you'll notice that $+\infty$ won't work!

2. If you already found that there is a **horizontal asymptote at** ∞, there **CANNOT** be a slant asymptote at $+\infty$ as well, (because that would mean that the function is approaching 2 lines at the same time, which is nonsense)! This simplifies your search a little bit! **HOWEVER**, there may be a horizontal asymptote at ∞ **AND** a slant asymptote at $-\infty$, so beware of this case! And similarly for $-\infty$ (i.e. horizontal asymptote at $-\infty$ implies there cannot be a slant asymptote at $-\infty$!)

3. **Periodic** functions **CANNOT** have slant asymptotes!